CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscate, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscate whose axis is b the axis a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Solving the equatians, $r\cos\theta+a\cos2\theta=0$ (strophoid), and $r=e^2\cos2\theta$ (lemniscate), we find they coincide when $\sin\theta=\sqrt{\tfrac{1}{2}}$ (1), or $\sin\theta=\sqrt{\dfrac{a^2-e^2}{2a^2-e^2}}$ (2).

(1) shows that they coincide at the origin for all values of a and e. We have to find the relation between the axes a and e which will make the curves tangent at the points determined by (2), provided those points are on both the leaves. Let $\phi=\angle$ made by the tangent at any point, with the radius vector drawn to that point. Then by the formula $\tan\phi=r\dfrac{d\theta}{dr}$.

Now for the lemniscate $r=\pm e\sqrt{\cos2\theta}$. $\dfrac{dr}{d\theta}=\dfrac{\mp e\sin2\theta}{\sqrt{\cos2\theta}}$.

$$\tan\phi=\pm e\sqrt{\cos2\theta}\left(\frac{\sqrt{\cos2\theta}}{\mp e\sin2\theta}\right)=\frac{2\sin^2\theta-1}{2\sin\theta\sqrt{1-\sin^2\theta}}\dots\dots\dots\dots\dots\dots(3).$$

For the strophoid $r=-a\cos2\theta/\cos\theta$.

$$dr/d\theta=-a(-2\cos\theta\sin2\theta+\cos2\theta\sin\theta)/\cos^2\theta.$$

$$\tan\phi=[a\cos^2\theta\cos2\theta]/[a\cos\theta(-2\cos\theta\sin2\theta+\cos2\theta\sin\theta)]$$

$$=[\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)]/[\sin\theta(2\sin^2\theta-3)]\dots\dots\dots\dots(4).$$

Now equate (3) and (4) and substitute from (1),

$$[\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)]/[\sin\theta(2\sin^2\theta-3)]=(2\sin^2\theta-1)/2\sin\theta\sqrt{1-\sin^2\theta}.$$

$$2\sin\theta(1-\sin^2\theta)(1-2\sin^2\theta)=\sin\theta(1-2\sin^2\theta)(3-2\sin^2\theta),$$

$$2\sqrt{\tfrac{1}{2}}(1-\tfrac{1}{2})(1-1)=\sqrt{\tfrac{1}{2}}(1-1)(3-1)\text{ or }0=0,$$

which shows that the curves are tangent at point ($\theta=\sin^{-1}\sqrt{\tfrac{1}{2}}$, $r=0$) for any value of a and e. Again substituting from (2),

$$2\sqrt{\frac{a^2-e^2}{2a^2-e^2}}\left(\frac{a^2}{2a^2-e^2}\right)\left(\frac{e^2}{2a^2-e^2}\right)=\sqrt{\frac{a^2-e^2}{2a^2-e^2}}\left(\frac{e^2}{2a^2-e^2}\right)\left(\frac{4a^2-e^2}{2a^2-e^2}\right).$$

This resolves into the three equations: $\sqrt{\dfrac{a^2-e^2}{2a^2-e^2}}=0$, whence $e=\pm a$..... (5);

$$\frac{e^2}{2a^2-e^2}=0, \text{ whence } e=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6);$$

$$\frac{2a^2}{2a^2-e^2}=\frac{4a^2-e^2}{2a^2-e^2}, \text{ whence } e=\pm a\sqrt{2}\ldots\ldots\ldots\ldots\ldots (7).$$

From (5) substituted in (2), $\sin\theta=0$. ∴ the curves are tangent at the extremity of the common axis, and the equations become,

$$r\cos\theta+a\cos2\theta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$r^2=a^2\cos2\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From (9) $r_1=\pm a\sqrt{\cos2\theta}$.

From (8) $r_2=\dfrac{-a\cos2\theta}{\cos\theta}=-a\sqrt{\cos2\theta}\sqrt{\dfrac{\cos2\theta}{1-\sin^2\theta}}=-a\sqrt{\cos2\theta}\sqrt{\dfrac{1-2\sin^2\theta}{1-\sin^2\theta}}$.

Since for any value of $\sin\theta$ numerically less than $\sqrt{\tfrac{1}{2}}$, $\sqrt{\dfrac{1-2\sin^2\theta}{1-\sin^2\theta}}$ is numerically less than 1, r_2 is then numerically less than r_1. But by tracing the curves the leaf of each is seen to be formed by values of θ determined by this limit. ∴ every point of the leaf of the strophoid lies within the lemniscate, and the former is in this case inscribed. From (6) equation of lemniscate becomes $r^2=0$, and the curve becomes a point. From (7) by substituting in (2) $\sin\theta=\sqrt{\dfrac{-a^2}{0}}$ an impossible value.

Accordingly the leaf of the strophoid can be inscribed in the leaf of the lemniscate when their axes are equal, and under no condition can the leaf of the lemniscate with an axis greater than 0 be inscribed in the leaf of the strophoid.

Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

[It will be seen that Professor Black's result does not realize the intention of the problem as given by the Proposer. However, even for the Proposer's reading of the problem, his solution seems to us to be defective in several points. We may give Professor Zerr's solution later. EDITOR.]